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Free-surface flow past oscillating singularities at resonant frequency

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This paper analyses the problem of a flow past an oscillating body moving with constant velocity, below and parallel to a free surface. Special attention is given to frequencies of oscillation in the neighbourhood of the critical frequency $\omega_c = 0{\cdot}25\,g/U$, where the classical linearized solution yields infinitely large wave amplitude. As a result both the lift and drag forces acting on the oscillating body at the resonant frequency are singular. It is demonstrated in the paper how this resonance is eliminated by considering higher-order free-surface effects, in particular the interaction between the first- and third-order terms. The resulting generalized solution yields finite wave amplitudes at the resonant frequency which are $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon \log \epsilon)$ for 2 and 3 dimensions respectively. Here ϵ is a measure of the singularity strength. It is also shown that inclusion of third-order terms causes a shift in the wavenumber and group velocity which eliminates the singularity in the lift and drag expressions at the resonant frequency. These results are illustrated by computing the lift and drag experienced by a submerged oscillating horizontal doublet in a uniform flow.

1. Introduction

The problem considered here is that of a flow past a body (or a pressure distribution) that moves at constant speed and simultaneously performs an oscillatory motion near the free-surface of a heavy fluid. Since a body or a travelling pressure patch can be represented approximately by a suitable distribution of singularities, we will refer explicitly herein to the case of submerged or free-surface singularities.

This problem has drawn some attention and has been investigated in the past because of its fundamental interest as well as for its applications in naval hydrodynamics in general and in ship motion in particular. So far the flow problem has been traditionally solved by linearization, i.e. under the assumption that the oscillatory motion is a small perturbation of the uniform flow, and consequently with neglect of nonlinear terms in the free-surface boundary condition. Thus the two-dimensional case has been treated by, among others, Haskind (1954), Wu (1957) and Debnath & Rosenblat (1969), while the three-dimensional flow has been investigated primarily by Eggers (1957), Hanaoka (1957), Becker (1958), Newman (1959), Tayler & Van den Driessche (1974) and Doctors (1978). In these previous studies it has been found that in water of infinite depth the velocity potential becomes unbounded at the critical resonant frequency $\omega_c = \omega'_c U'/g = 0{\cdot}25$, where ω' is the frequency of oscillation, U' is the uniform translatory velocity and g is the acceleration of gravity.† In the two-

† A recent work dealing with the numerical solution of the same problem (Euvrard *et al.* 1977) emphasizes the inability of numerical methods to cope with resonant conditions.

dimensional case, for frequencies smaller than ω_c', there exists a system of three distinct waves that propagate downstream from the singularity, and one wave that propagates upstream, whereas for $\omega' > \omega_c'$ only two downstream waves are left. The group velocity in still water of the two waves that disappear at $\omega' > \omega_c'$ tends to the translation velocity U' as ω' approaches ω_c' from below. The resonance phenomenon is thus attributed to the inability of these waves to transfer away the energy imparted by the oscillating body to the fluid. The picture is essentially the same in the case of three-dimensional flows for the transverse wave system. It has been shown in the past, and also in the sequel here, that the amplitude of the resonant waves is of order $(\omega_c' - \omega')^{-\frac{1}{2}}$ for two-dimensional flow and of order $\ln(\omega_c' - \omega')$ in the three-dimensional case.

It is well known by now that the linearized solution ceases to be valid at near-resonant conditions, but nevertheless no attempt has been made so far to remove this resonance by taking account of nonlinear free-surface effects. In a recent work (Dagan & Miloh 1981), we have investigated the effect of a steady non-uniform flow, associated with the singularity, upon the resonance. One of the main findings of this work was that resonance is removed in the solution of the linearized equations, but with variable coefficients related to the non-uniform flow maintained in the free-surface equation, only for the case in which the steady component is due to an isolated vortex or a lifting line. Although this case is of interest in some applications, the general problem which involves other types of singularities and of purely oscillatory motion was not solved.

The aim of the present work is to solve this general problem of resonance removal for both two- and three-dimensional flows. We use a perturbation expansion in the amplitude ϵ' of the oscillatory singularity, and we shall show that the perturbation series becomes non-uniform near ω_c' when terms of third order in ϵ' are retained in the expansion. By a proper uniformization procedure, similar to co-ordinate straining, a finite solution which is valid at ω_c' is subsequently obtained. The free-surface third-order nonlinear terms cause a shift of the wavenumber of the free waves, and consequently their group velocity does not tend to U' when $\omega' \to \omega_c'$, which essentially explains the removal of resonance. Furthermore, we shall show that the velocity potential, of order ϵ' for ω' sufficiently far from ω_c', becomes of order $\epsilon'^{\frac{1}{2}}$ and $\epsilon' \ln \epsilon'$ at $\omega' = \omega_c'$ in the two- and three-dimensional cases respectively. Finally, we shall illustrate the results by evaluating the forces acting on a dipole singularity at ω' close to ω_c'.

The two-dimensional problem is considered in detail here since closed-form results can be obtained more easily for the planar case. Furthermore, the case of three-dimensional flows can be solved along the same lines, although some additional algebraic difficulties are present. The main results of the analysis of three-dimensional flows are briefly sketched in §9.

2. Mathematical statement of the problem and perturbation expansion

We consider the inviscid flow of a heavy fluid of infinite depth. Variables are made dimensionless with respect to U' and U'^2/g as velocity and length scales, respectively, where $U' > 0$ is the constant horizontal velocity of the moving singularity. Let (x, y) be a Cartesian system moving with the singularity, with y vertically upwards while

$y = 0$ is the equation of the unperturbed free surface. Let $\Phi(x, y, t)$ be the velocity potential, which is decomposed as follows:

$$\Phi(x, y, t) = -x + \phi(x, y, t). \tag{1}$$

Here ϕ denotes the perturbation potential, which satisfies the following exact equations:

$$\nabla^2\phi = 0, \quad (y \leqslant \eta), \tag{2}$$

$$\eta(x, t) = -\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} - \tfrac{1}{2}\nabla\phi . \nabla\phi, \quad (y = \eta) \tag{3}$$

$$\frac{\partial^2\phi}{\partial t^2} - 2\frac{\partial^2\phi}{\partial t\,\partial x} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y} + 2\nabla\phi . \frac{\partial}{\partial t}\nabla\phi - 2\nabla\phi . \frac{\partial}{\partial x}\nabla\phi + \tfrac{1}{2}\nabla\phi . \nabla(\nabla\phi . \nabla\phi) = 0, \quad (y = \eta), \tag{4}$$

where $y = \eta(x, t)$ is the equation of the free surface. Equation (4) can be obtained by elimination of η from the Bernoulli equation (3) and the kinematic free-surface boundary condition (see e.g. Wehausen & Laitone 1960). For the sake of convenience (4) is rewritten as

$$L_1(\phi) + L_2(\phi, \phi) + L_3(\phi, \phi, \phi) = 0, \tag{5}$$

where the linear, quadratic and third-order operators L_j $(j = 1, 2, 3)$ are given by

$$L_1(\alpha) = \frac{\partial^2\alpha}{\partial t^2} - \frac{\partial^2\alpha}{\partial t\,\partial x} + \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial\alpha}{\partial y}, \tag{6}$$

$$L_2(\alpha, \beta) = 2\nabla\alpha . \frac{\partial}{\partial t}\nabla\beta - 2\nabla\alpha . \frac{\partial}{\partial x}\nabla\beta, \tag{7}$$

$$L_3(\alpha, \beta, \gamma) = \sum_{n=1}^{2}\sum_{m=1}^{2} \frac{\partial\alpha}{\partial x_n}\frac{\partial\beta}{\partial x_m}\frac{\partial^2\gamma}{\partial x_n\,\partial x_m} \quad (x_1 \equiv x,\ x_2 \equiv y). \tag{8}$$

We now expand the disturbance potential $\phi(1)$ in a perturbation power series in ϵ, a small dimensionless parameter that characterizes the singularity strength. Thus

$$\phi(x, y, t) = \epsilon\phi_1(x, y, t) + \epsilon^2\phi_2(x, y, t) + \epsilon^3\phi_3(x, y, t) + \dots. \tag{9}$$

Substituting (9) into (2) yields

$$\nabla^2\phi_j(x, y, t) = 0 \quad (y \leqslant 0;\ j = 1, 2, \dots), \tag{10}$$

while (3) yields, after a Taylor expansion near $y = 0$,

$$\eta_1(x, t) = -\frac{\partial\phi_1}{\partial t} + \frac{\partial\phi_1}{\partial x} \quad (y = 0), \tag{11}$$

$$\eta_2(x, t) = -\frac{\partial\phi_2}{\partial t} + \frac{\partial\phi_2}{\partial x} + \eta_1\frac{\partial}{\partial y}\left(-\frac{\partial\phi_1}{\partial t} + \frac{\partial\phi_1}{\partial x}\right) - \tfrac{1}{2}\nabla\phi_1 . \nabla\phi_1 \quad (y = 0). \tag{12}$$

Carrying out a similar expansion of (4) gives, up to third-order terms,

$$L_1(\phi_1) = 0 \quad (y = 0), \tag{13}$$

$$L_1(\phi_2) = -L_2(\phi_1, \phi_1) - \eta_1\frac{\partial}{\partial y}L_1(\phi_1) \quad (y = 0), \tag{14}$$

$$L_1(\phi_3) = -L_3(\phi_1, \phi_1, \phi_1) - \eta_2\frac{\partial L_1(\phi_1)}{\partial y} - \tfrac{1}{2}(\eta_1)^2\frac{\partial^2 L_1(\phi_1)}{\partial y^2} - \eta_1\frac{\partial L_2(\phi_1, \phi_1)}{\partial y} - \eta_1\frac{\partial L_1(\phi_2)}{\partial y} - L_2(\phi_1, \phi_2) - L_2(\phi_2, \phi_1) \quad (y = 0). \tag{15}$$

The solution of the problem by the perturbation-expansion scheme involves the determination of harmonic functions ϕ_j defined in the half-plane $y \leqslant 0$, subject to the boundary conditions (11)–(15) on $y = 0$. We now consider explicitly oscillating singularities by separating ϕ_1 in the following manner:

$$\phi_1(x,y,t) = \varphi_1(x,y,t) + 2\psi_1(x,y)\cos\omega t, \tag{16}$$

where $\psi_1(x,y)$ is a singular function satisfying the condition

$$\psi_1(x,0) = 0 \tag{17}$$

on the unperturbed free surface, and $\varphi_1(x,y,t)$ is regular for $y \leqslant 0$. We seek periodic quasi-steady solutions for ϕ_j in the form

$$\phi_j(x,y,t) = \sum_{m=-\infty}^{\infty} \varphi_{j,m}(x,y)\, e^{im\omega t} + 2\psi_1(x,y)\cos\omega t; \quad \varphi_{j,-m} = \bar{\varphi}_{j,m}, \quad \eta_{j,-m} = \bar{\eta}_{j,m}, \tag{18}$$

where $\bar{\varphi}$ and $\bar{\eta}$ stand for the complex conjugates of φ and η.

To account for the initial conditions or equivalently for radiation conditions, we shall use the method suggested by Lighthill (1960); namely we multiply the right-hand side of (18) by $\exp(\mu t)$ and seek the limit $\mu \to 0$ as the solution of our problem (this is equivalent to using an artificial viscosity in the manner suggested by Rayleigh). With the notation of (18), ψ_1 (16) can be rewritten as

$$2\psi_1 \cos\omega t = \psi_{1,1}(x,y)\, e^{i\omega t} + \psi_{1,-1}(x,y)\, e^{-i\omega t}; \quad \psi_1 \equiv \psi_{1,1} \equiv \psi_{1,-1}. \tag{19}$$

To derive the equations satisfied by the unknown functions $\varphi_{j,m}$, we substitute (18) and (19) into (10)–(15) to obtain for $j = 1,2,3$ (after collecting terms with same frequency)

$$\nabla^2 \varphi_{j,m} = 0 \quad (y \leqslant 0); \tag{20}$$

first-order

$$L_1(\varphi_{1,1}) = -L_1(\psi_{1,1}) = -\frac{\partial \psi_{1,1}}{\partial y} = p_{1,1}(x) \quad (y=0), \tag{21}$$

$$\eta_{1,1} = -i\omega\varphi_{1,1} + \frac{\partial \varphi_{1,1}}{\partial x} \quad (y=0), \tag{22}$$

$$\varphi_{1,m} \equiv 0 \quad (m \neq 1, -1); \tag{23}$$

second-order

$$L_1(\varphi_{2,2}) = -L_2(\varphi_{1,1}, \varphi_{1,1}) - \eta_{1,1}\frac{\partial L_1(\varphi_{1,1})}{\partial y} = p_{2,2}(x) \quad (y=0), \tag{24}$$

$$L_1(\varphi_{2,0}) = -\eta_{1,-1}\frac{\partial L_1(\varphi_{1,1})}{\partial y} - \eta_{1,1}\frac{\partial L_1(\varphi_{1,-1})}{\partial y} - L_2(\varphi_{1,1}, \varphi_{1,-1}) - L_2(\varphi_{1,-1}, \varphi_{1,1}) = p_{2,0}(x) \quad (y=0), \tag{25}$$

$$\eta_{2,2} = -2i\omega\varphi_{2,2} + \frac{\partial \varphi_{2,2}}{\partial x} - \omega^2\varphi_{1,1}\frac{\partial \varphi_{1,1}}{\partial y} - \eta_{1,1}\frac{\partial}{\partial y}\left(-i\omega\varphi_{1,1} + \frac{\partial \varphi_{1,1}}{\partial x}\right) - \tfrac{1}{2}\nabla\varphi_{1,1}\cdot\nabla\varphi_{1,1} \quad (y=0), \tag{26}$$

$$\eta_{20} = i\omega\eta_{1,1}\frac{\partial \varphi_{1,-1}}{\partial y} - i\omega\eta_{1,-1}\frac{\partial \varphi_{1,1}}{\partial y} + \eta_{1,1}\frac{\partial^2 \varphi_{1,-1}}{\partial x\,\partial y} + \eta_{1,-1}\frac{\partial^2 \varphi_{1,1}}{\partial x\,\partial y} - \nabla\varphi_{1,1}\cdot\nabla\varphi_{1,-1} + \frac{\partial \varphi_{2,0}}{\partial x} \quad (y=0), \tag{27}$$

$$\varphi_{2,m} \equiv 0 \quad (m \neq -2, 0, 2); \tag{28}$$

third-order

$$
\begin{aligned}
L_1(\varphi_{3,1}) = & -L_3(\varphi_{1,-1}, \varphi_{1,1}, \varphi_{1,1}) - L_3(\varphi_{1,1}, \varphi_{1,-1}, \varphi_{1,1}) - L_3(\varphi_{1,1}, \varphi_{1,1}, \varphi_{1,-1}) \\
& - \eta_{2,2} \frac{\partial L_1(\varphi_{1,-1})}{\partial y} - \tfrac{1}{2}(\eta_{1,1})^2 \frac{\partial^2 L_1(\varphi_{1,-1})}{\partial y^2} - \eta_{1,1}\eta_{1,-1} \frac{\partial^2 L_1(\varphi_{1,1})}{\partial y^2} \\
& - \eta_{1,1} \frac{\partial}{\partial y}[L_2(\varphi_{1,1}, \varphi_{1,-1}) + L_2(\varphi_{1,-1}, \varphi_{1,1})] - \eta_{1,-1} \frac{\partial}{\partial y} L_2(\varphi_{1,1}, \varphi_{1,1}) \\
& - \eta_{2,0} \frac{\partial L_1(\varphi_{1,1})}{\partial y} - \eta_{1,1} \frac{\partial L_1(\varphi_{2,0})}{\partial y} - L_2(\varphi_{1,-1}, \varphi_{2,2}) - L_2(\varphi_{2,0}, \varphi_{1,1}) \\
& - L_2(\varphi_{2,2}, \varphi_{1,-1}) - L_2(\varphi_{1,1}, \varphi_{2,0}) = p_{3,1}(x) \quad (y = 0),
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
L_1(\varphi_{3,3}) = & -\eta_{2,2} \frac{\partial L_1(\varphi_{1,1})}{\partial y} - \tfrac{1}{2}\eta_{1,1}^2 \frac{\partial L_1(\varphi_{1,1})}{\partial y} - \eta_{1,1} \frac{\partial L_2(\varphi_{1,1}, \varphi_{1,1})}{\partial y} - \eta_{1,1} \frac{\partial L_1(\varphi_{2,2})}{\partial y} \\
& - L_2(\varphi_{1,1}, \varphi_{2,2}) - L_2(\varphi_{2,2}, \varphi_{1,1}) - L_3(\varphi_{1,1}, \varphi_{1,1}, \varphi_{1,1}) = p_{3,3}(x) \quad (y = 0),
\end{aligned} \tag{30}
$$

$$\varphi_{3\,m} \equiv 0 \quad (m \neq 1, -1, 3, -3). \tag{31}$$

Equations (21)–(31) exhaust the boundary conditions satisfied by $\varphi_{1,1}$, $\varphi_{2,2}$, $\varphi_{2,0}$, $\varphi_{3,1}$ and $\varphi_{3,3}$, the only terms different from zero up to third order. We have assumed tacitly that higher-order terms than ϕ_1 do not comprise singular potentials. This is not necessarily the case, but, as shown in the sequel, their presence does not affect our conclusions regarding the resonance removal. In any case, such terms can be added at will to the right-hand sides of (21), (24), (25, (21)–(30).

The operators L_1, L_2 and L_3 appearing in (21)–(31) are the same as in (6)–(8), after differentiating with respect to time and deleting the exponential time-dependent term. Thus $\partial\phi_j/\partial t$, for example, has to be replaced by $\Sigma_{n=-\infty}^{\infty}(im\omega+\mu)\,\varphi_{j,m}$ in L_1 (6) and L_2 (7).

Summarizing this section, the problem of a free-surface flow past on oscillatory singularity has been reduced, by employing a perturbation expansion, to the determination of the potentials $\varphi_{j,m}$, which satisfy linear but inhomogeneous boundary conditions on $y = 0$. The right-hand-side terms $p_{j,m}$ in (24), (25), (29) and (30) are all functions of fewer than m terms of the potential, and may be thus considered as inhomogeneous source terms.

3. Solution by Fourier transforms

To solve (20)–(31) we make use of Fourier transforms of the regular potentials $\varphi_{j,m}$, defined by

$$\tilde{\varphi}_{j,m}(k, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \varphi_{j,m}(x, 0)\, e^{ikx} e^{|k|y}\, dx, \tag{32}$$

$$\varphi_{j,m}(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \tilde{\varphi}_{j,m}(k, 0)\, e^{-ikx} e^{|k|y}\, dk, \tag{33}$$

such that the Laplace equations (20) are satisfied. Taking now the Fourier transform of (21), (24), (25), (29) and (30) yields

$$A_{j,m}(k, \omega)\, \tilde{\varphi}_{j,m}(k, 0) = \tilde{p}_{j,m}(k) \quad (j = 1, m = 1; j = 2, m = 2; j = 2, m = 0; j = 3, m = 1; j = 3, m = 3). \tag{34}$$

The polynomials $A_{j,m}$ are given by the following expressions, based on (6) and (18):

$$A_{j,m}(k,\omega) = -m^2\omega^2 - 2m\omega k - k^2 + |k| - 2i\omega m\mu + 2imk\mu, \tag{35}$$

for any j and with μ a positive vanishingly small parameter. The set (34) gives $\varphi_{j,m}$ for $y = 0$, and $\phi_{j,m}(x,y)$ can thus be obtained subsequently by inversion (33).

Before embarking on the detailed analysis of the various $\varphi_{j,m}$ we shall briefly discuss the simple, but illuminating case, of a harmonic wave.

4. Solution for a harmonic wave

Let us consider first the case $\psi_1 \equiv 0$, or $p_{1,1} \equiv 0$, i.e. no singularity, while the first-order potential is given by

$$\varphi_{1,1} = \epsilon e^{-ikx} e^{|k|y}. \tag{36}$$

As a matter of fact these potentials represent in a fixed frame the usual Stokes waves, and the remaining terms of the expansion are its well-known higher-order approximations. It is still instructive to develop these expressions in the moving frame by following the present procedure.

The wave profile, according to (11) and (22), is given by

$$\eta_1(x,t) = \eta_{1,1} e^{i\omega t} + \eta_{1,-1} e^{-i\omega t} = -2\epsilon(k+\omega)\sin(kx - \omega t). \tag{37}$$

Substitution of (36) into (34), with $p_{1,1} \equiv 0$ and $\mu = 0$, yields the well-known dispersion relation

$$A_{1,1}(k,\omega) = -\omega^2 - 2\omega k + |k| - k^2 = 0. \tag{38}$$

The function $A_{1,1}(k,\omega)$ is depicted schematically in figure 1. The four roots of (38) are given by

$$k_1^{(1)}, k_1^{(2)} = \tfrac{1}{2}[1 - 2\omega \pm (1-4\omega)^{\frac{1}{2}}], \tag{39}$$

$$k_1^{(3)}, k_1^{(4)} = \tfrac{1}{2}[-1 - 2\omega \pm (1+4\omega)^{\frac{1}{2}}]; \tag{40}$$

and in the neighbourhood of $\omega = \omega_c = 0{\cdot}25$, (39) becomes

$$k_1^{(1)}, k_1^{(2)} = k_c + \delta\omega \pm \delta\omega^{\frac{1}{2}}, \tag{41}$$

with

$$k_c = \omega_c = \tfrac{1}{4}, \quad \delta\omega = \omega_c - \omega. \tag{42}$$

The group velocities of the first two waves in a fixed frame are given by

$$c_g^{(1)}, c_g^{(2)} = -1 + (dk/d\omega)^{-1} = -1 \mp \frac{2\delta\omega^{\frac{1}{2}}}{1 \pm 2\delta\omega^{\frac{1}{2}}}. \tag{43}$$

At $\omega = \omega_c (\delta\omega = 0)$ the group velocity is equal to the current velocity; i.e. the wave energy flux with respect to the moving frame is zero.

Substituting $\varphi_{1,1}$ (36) with $\varphi_{1,-1} = \bar{\varphi}_{1,1}$ into $p_{2,2}(x)$ (24) and $p_{2,0}$ (25), and by using (6)–(8), it is immediately found that

$$p_{2,2}(x) \equiv p_{2,-2}(x) \equiv p_{2,0}(x) \equiv 0, \tag{44}$$

and consequently

$$\varphi_{2,2} \equiv \varphi_{2,-2} \equiv \varphi_{2,0} \equiv 0. \tag{45}$$

Furthermore, substitution of (36) and (45) into (29) and (30) shows that $\varphi_{3,3} \equiv 0$, and that the only surviving terms of $p_{3,1}$ are

$$\begin{aligned} p_{3,1} &= -L_3(\varphi_{1,-1}, \varphi_{1,1}, \varphi_{1,1}) - L_3(\varphi_{1,1}, \varphi_{1,-1}, \varphi_{1,1}) - L_3(\varphi_{1,1}, \varphi_{1,1}, \varphi_{1,-1}) \\ &= -4\epsilon^3 k^4 e^{-ikx}. \end{aligned} \tag{46}$$

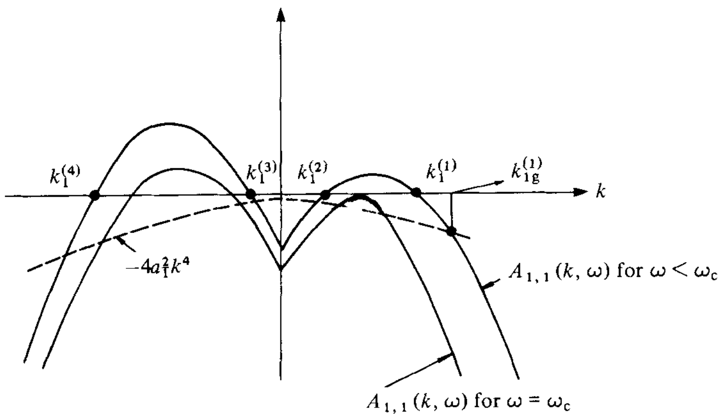

FIGURE 1. Graphical representation of the solution of (38) and (51).

It is thus seen that the perturbation scheme breaks down at third order. Indeed, for

$$\varphi_{3,1} = a_3\, e^{-ikx}\, e^{|k|y} \tag{47}$$

(29) gives for the unknown amplitude a_3

$$A_{1,1}(k,\omega)\, a_3 = -4\epsilon^3 k^4, \tag{48}$$

which by (38) implies $a_3 \to \infty$.

The appearance of the secular term in the third-order approximation can be attributed to an amplitude related wavenumber shift. Indeed, if we collect the first- and third-order terms in a generalized expansion of the type

$$\varphi_{1,1g} = \epsilon\, e^{-ik_g(\epsilon)x}\, e^{|k_g(\epsilon)|y}, \quad \varphi_{1,-1g} = \epsilon\, e^{ik_g x}\, e^{|k_g|y}, \tag{49}$$

then k_g is determined by the equations

$$L_1(\varphi_{1,1g}) = p_{3,1}(x) \quad (y = 0). \tag{50}$$

Equations (49) and (50) yield, by using (38) and (46), the following generalized dispersion equation:

$$A_{1,1g}(k_g,\omega) = -4\epsilon^2 k_g^4. \tag{51}$$

The right-hand side of (51) is represented in fig. 1, and the four roots of (51) are shown there schematically. For sufficiently small ϵ that satisfies the requirement

$$\epsilon^3 k^3 \frac{dA_{1,1}(k,\omega)}{dk} \ll 1 \tag{52}$$

the roots of (51) are close to those of (38) and can be computed by taking $k_g = k_1^{(j)}$ ($j = 1,2,3,4$) in the right-hand side of (51).

In particular, for the two roots (39) and for ω close to ω_c we obtain

$$\begin{aligned} k_{1g}^{(1)}, k_{1g}^{(2)} &= \tfrac{1}{2}[1 - 2\omega \pm (1 - 4\omega + \tfrac{1}{16}\epsilon^2)^{\frac{1}{2}}] \\ &= k_c + \delta\omega \pm (\delta\omega + \tfrac{1}{64}\epsilon^2)^{\frac{1}{2}}. \end{aligned} \tag{53}$$

It is seen that for $\delta\omega = 0$ the wavenumber of the free-waves is shifted by $\frac{1}{8}|\epsilon|$, and the group velocity (43) is no longer equal to the current velocity. These rather simple results provide the key to the resonance removal in the case of a moving singularity, which is the main theme of this study. As we shall show in the following sections, the algebra then becomes more involved owing to the presence of the singular potential $\psi_{1,1}$ in (21). The final result, however, is that the wave energy is carried away from the singularity and the wave amplitude remains finite for $\omega \leqslant \omega_c$.

The role of third-order interactions of free waves in resonant conditions has been investigated thoroughly in the past (see e.g. Phillips 1969, p. 135). Some results for steady waves generated by a ship have also been obtained by Newman (1971).

5. The first-order solution

We return now to the case of oscillating singularities, and present the solution of the first-order problem. This is a classical problem which has been solved previously (see § 1), and here we reproduce the results briefly for the sake of completeness. Thus, (21) and (34) yield

$$\tilde{\varphi}_{1,1}(k,0) = \frac{\tilde{p}_{1,1}(k)}{A_{1,1}(k,\omega)}. \tag{54}$$

The quadratic form $A_{1,1}$ (38) can be rewritten as

$$A_{1,1}(k,\omega) = -(k-k_1^{(1)})(k-k_1^{(2)}) \quad (k>0), \tag{55}$$

$$A_{1,1}(k,\omega) = -(k-k_1^{(3)})(k-k_1^{(4)}) \quad (k<0), \tag{56}$$

where the four roots $k_1^{(j)}$ are given by (39) and (40). If the small parameter μ is kept in the expression for $A_{1,1}$, the locations of these roots on the inversion path in the complex k-plane is determined completely, as shown in figure 2(a). The potential $\varphi_{1,1}(x,y)$ is obtained by inverting $\tilde{\varphi}_{1,1}$. Thus by accounting for the semi-residues at $k_1^{(j)}$, we have

$$\begin{aligned}\varphi_{1,1}(x,y) = {} & \frac{1}{(2\pi)^{\frac{1}{2}}} \fint_{-\infty}^{\infty} \frac{\tilde{p}_{1,1}(k)\exp\{-ikx\}\exp\{|k|y\}}{A_{1,1}(k,\omega)}\,dk \\ & + i(\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{j=1,3,4} \frac{\tilde{p}_{1,1}(k_1^{(j)})\exp\{-ik_1^{(j)}x\}\exp\{|k_1^{(j)}|y\}}{[A_{1,1}(k,\omega)/(k-k_1^{(j)})]_{k=k_1^{(j)}}} \\ & - i(\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{\tilde{p}_{1,1}(k_1^{(2)})\exp\{-ik_1^{(2)}x\}\exp\{|k_1^{(2)}|y\}}{k_1^{(1)}-k_1^{(2)}} \quad (\omega \leqslant \omega_c),\end{aligned} \tag{57}$$

where it has been assumed that $\tilde{p}_{1,1}$ (54) is regular on the real k-axis. The integral term in (57) stands for a Cauchy principal value at the four poles of figure 2(a). We are interested here only in the resonant terms of $\varphi_{1,1}$, i.e. those that become unbounded as $\omega \to \omega_c$. It is readily seen that these terms are associated with the semi-residues at $k_1^{(1)}$, $k_1^{(2)}$, and for fixed (x,y) $\varphi_{1,1}$ (57) can be rewritten as

$$\varphi_{1,1}^{*}(x,y) = -i(2\pi)^{\frac{1}{2}} \frac{\tilde{p}_{1,1}(k_c)\,e^{ik_c x}\,e^{k_c y}}{\delta\omega^{\frac{1}{2}}} \quad (\omega \leqslant \omega_c), \tag{58}$$

where the remaining terms of (57) are finite as $\delta\omega \to 0$ and $k_c = 0{\cdot}25$. Similarly we have

$$\varphi_{1,-1}^{*}(x,y) = i(2\pi)^{\frac{1}{2}} \frac{\tilde{p}_{1,1}(-k_c)\,e^{-ik_c x}\,e^{k_c y}}{\delta\omega^{\frac{1}{2}}} \quad (\omega \leqslant \omega_c), \tag{59}$$

since $\tilde{p}_{1,1}(k) = \bar{\tilde{p}}_{1,1}(-k)$. Hence we arrive at the final expression for the first-order velocity potential defined in (18), for ω close to ω_c,

$$\phi_1(x,y,t) = \frac{i(2\pi)^{\frac{1}{2}}}{\delta\omega^{\frac{1}{2}}} e^{k_c y}\,[\tilde{p}_{1,1}(-k_c)\,e^{-ik_c x - i\omega t} - \tilde{p}_{1,1}(k_c)\,e^{ik_c x + i\omega t}] + O(1). \tag{60}$$

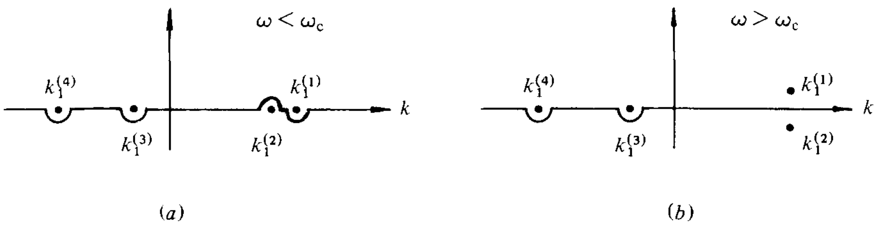

FIGURE 2. Location of roots of $A_{1,1}(k, \omega) = 0$ on the inversion path in the complex k-plane: (a) for $\omega < \omega_c$; (b) for $\omega > \omega_c$.

Also, by (19) and (21),

$$\tilde{p}_{1,1}(k) = -\widetilde{\frac{\partial \psi_{1,1}(x,y)}{\partial y}}\bigg|_{y=0}. \tag{61}$$

Similar expressions can be obtained for $|x| \to \infty$. Then, the residues at $k_1^{(1)}$ or $k_1^{(2)}$, depending on whether $x \to \infty$ or $x \to -\infty$, contribute to the resonant term of $\varphi_{1,1}$. Thus

$$\phi_1(x,y,t) = \frac{i(2\pi)^{\frac{1}{2}}}{\delta\omega^{\frac{1}{2}}} \exp\{k_1^{(1)}y\}[\tilde{p}_{1,1}(-k_1^{(1)})\exp\{-ik_1^{(1)}x - i\omega t\} - \tilde{p}_{1,1}(k_1^{(1)})\exp\{ik_1^{(1)}x + i\omega t\}] + O(1) \quad (x \to \infty,\ \omega \leqslant \omega_c), \tag{62}$$

and there is a similar expression, with $k_1^{(1)}$ replaced by $k_1^{(2)}$, for $x \to -\infty$.

In the case $\omega > \omega_c$, the two resonant poles $k_1^{(1)}$ and $k_1^{(2)}$ depart from the real axis (figure 2b), and the polar contributions in (62) vanish. Hence $\epsilon\phi_1$ is not resonant as long as ω approaches ω_c from the right, i.e. $\omega > \omega_c$, and has an infinite sharp discontinuity at $\omega = \omega_c$. Finally, the amplitude of the resonant term is $O[\epsilon/\delta\omega^{\frac{1}{2}}]$ for $\omega \leqslant \omega_c$.

6. The higher-order solutions

The Fourier transforms of the higher-order terms $\tilde{\varphi}_{j,m}$ are given by (34) and (35). Unlike $\tilde{p}_{1,1}$ (61), the nonlinear terms $\tilde{p}_{j,m}$(34) are expressed with the aid of convolution integrals, and their analysis, for arbitrary ω, is quite tedious. In the present work we are interested, however, only in the behaviour of $\varphi_{j,m}$ in the neighbourhood of $\omega = \omega_{cr}$, and more precisely in the most singular contributions to $\varphi_{j,m}$ as $\delta\omega \to 0$. These contributions can be obtained quite easily by extracting from the Fourier transforms of $p_{j,m}$ the semi-residues at the two poles neighbouring k_{cr} whenever they appear on the integration path in the transform plane. Such an analysis, whose details are not given here, shows that the highest resonant contributions of the second-order terms $\epsilon^2\varphi_{2,2}$ and $\epsilon^2\varphi_{2,0}$ are of order $\epsilon^2/\delta\omega^{\frac{1}{2}}$ for $\delta\omega \to 0$. Since the first-order solution $\epsilon\phi_1$ has been shown in §5 to be of order $\epsilon/\delta\omega^{\frac{1}{2}}$ it is seen that the quadratic term is weaker than the linearized solution and is asymptotic to $\epsilon\phi_1$ for any ω. The analysis of the third-order term $\varphi_{3,3}$ (30) reveals that its most singular contribution is of order $\epsilon^3/\delta\omega^{\frac{3}{2}}$ for $\delta\omega \to 0$, and this is found to be weaker than that of the remaining third-order term $\varphi_{3,1}$ (29), which is the only one investigated in some detail in the sequel.

Among the various terms making up $p_{3,1}$ (29), the most resonant contributions stem from the first three terms on the right-hand side of (29), exactly as in the case of the harmonic wave (46). Denoting this part by $p_{3,1}^*$ we have from (29) that

$$\widetilde{p_{3,1}^*}(k) = -\tilde{L}_3(\varphi_{1,-1}, \varphi_{1,1}, \varphi_{1,1}) - \tilde{L}_3(\varphi_{1,1}, \varphi_{1,-1}, \varphi_{1,1}) - \tilde{L}_3(\varphi_{1,1}, \varphi_{1,1}, \varphi_{1,-1})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_{3,1}^*(k-\lambda, \lambda-\nu, \nu)\,\tilde{\varphi}_{1,1}(\nu,0)\,\tilde{\varphi}_{1,1}(\lambda-\nu,0)\,\tilde{\varphi}_{1,-1}(k-\lambda,0)\,d\lambda\,d\nu, \tag{63}$$

where, by (8) and the faltung theorem

$$a^*_{3,1}(k-\lambda,\lambda-\nu,\nu) = \frac{1}{\pi}\{(\omega+\nu)[|\lambda-\nu|+|k-\lambda|](k-\nu)[(\lambda-\nu)(k-\lambda) - |\lambda-\nu||k-\lambda|] - (k-\lambda-\omega)[2\omega\lambda^2|\lambda-\nu| - 2\omega\nu|\nu|(\lambda-\nu) + |\nu|(\nu-|\nu|)(\lambda-\nu)(|\lambda-\nu|+|\nu|)] - [(\lambda-\nu)\nu^2(k-\lambda) + \tfrac{1}{2}(\lambda-\nu)\nu(k-\lambda)^2 - |\lambda-\nu||\nu|(k-\lambda)\nu - (\lambda-\nu)|\nu||k-\lambda|\nu - (\lambda-\nu)|\nu|(k-\lambda)|k-\lambda| + |\lambda-\nu|\nu^2|k-\lambda| + \tfrac{1}{2}|\lambda-\nu||\nu|(k-\lambda)^2]\}. \tag{64}$$

Furthermore, by substituting $\tilde{\varphi}_{1,1}$ (54) into (63) we obtain

$$\tilde{p}^*_{3,1}(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{b^*_{3,1}(k-\lambda,\lambda-\nu,\nu)}{A_{1,1}(\nu,\omega)\,A_{1,1}(\lambda-\nu,\omega)\,A_{1,-1}(k-\lambda,\omega)}\,d\lambda\,d\nu, \tag{65}$$

where, by (61),

$$b^*_{3,1}(k-\lambda,\lambda-\nu,\nu) = a^*_{3,1}(k-\lambda,\lambda-\nu,\nu)\,\tilde{p}_{1,1}(\nu)\,\tilde{p}_{1,1}(\lambda-\nu)\,\tilde{p}_{1,1}(k-\lambda). \tag{66}$$

Repeated extraction of the semi-residues of the coalescing poles on the integration paths in the ν- and λ-planes of $\tilde{p}^*_{3,1}$ (65) can be shown to yield the following most-singular contribution:

$$\tilde{p}^*_{3,1}(k) = \frac{\pi^2}{\delta\omega}\frac{b^*_{3,1}(-k_c,k_c,k_c)}{A_{1,1}(k,\omega)} \tag{67}$$

where $b^*_{3,1}(-k_c,k_c,k_c)$ (66) is regular and generally different from zero. Finally, the inversion of $\tilde{\varphi}_{3,1}(k,0)$ with $\tilde{p}_{3,1}$ replaced by $\tilde{p}^*_{3,1}(k)$ (67) yields for the most-singular term of $\varphi_{3,1}$,

$$\varphi^*_{3,1}(x,y) = \frac{\pi^{\frac{3}{2}}b^*_{3,1}(-k_c,k_c,k_c)}{2^{\frac{1}{2}}\delta\omega}\int_{-\infty}^{\infty}\frac{e^{-ikx}\,e^{|k|y}}{[A_{1,1}(k,\omega)]^2}dk. \tag{68}$$

The far free waves associated with $\varphi^*_{3,1}$ are therefore of the type

$$\varphi^*_{3,1}(x,y) = \frac{i2^{\frac{1}{2}}\pi^{\frac{5}{2}}b^*_{3,1}(-k_c,k_c,k_c)}{\delta\omega}\frac{\partial}{\partial k}\left[\frac{e^{-ikx}\,e^{|k|y}}{(k-k_1^{(2)})^2}\right]_{k=k_1^{(1)}} \quad (x\to\infty), \tag{69}$$

and similarly for $x\to-\infty$, with $k_1^{(2)}$ replaced by $k_1^{(1)}$.

It is seen that $\varphi^*_{3,1}$ consists of two terms: one of type $e^{-ik_c x}/\delta\omega^{\frac{5}{2}}$ and the other of type $xe^{-k_c x}/\delta\omega^2$. If we compare $\epsilon^3\phi_3 \sim \epsilon^3\varphi^*_{3,1} \sim \epsilon^3/\delta\omega^{\frac{5}{2}}$ stemming from the terms of first type with the corresponding resonant first-order term $\epsilon\phi_1 \sim \epsilon/\delta\omega^{\frac{1}{2}}$ (§5), it is found that the perturbation series is no longer uniform for fixed x, y, ϵ and $\delta\omega\to 0$. Indeed,

$$\epsilon^3\phi_3/\epsilon\phi_1 \sim \epsilon^2/\delta\omega^2, \quad (\omega \leqslant \omega_c) \tag{70}$$

implying that the perturbation series is uniform only if $\delta\omega \sim |\epsilon|^\alpha$, $\alpha < 1$. Hence in the (ϵ,ω)-plane the perturbation series is not uniform in the zone between the lines $\omega = \omega_c$ and $\omega = \omega_c - \beta|\epsilon|$, where β is finite.

Another disturbing finding related to terms of the second type is that $\epsilon^3\varphi^*_{3,1}$ becomes unbounded like $\epsilon^3|x|/\delta\omega^2$ for $|x|\to\infty$ and for any ϵ and $\delta\omega$. This is a secular term which recalls the breaking down of the perturbation expansion for a harmonic wave. Since the third-order term becomes dominant in the zone of non-uniformity, it has to be kept in the first approximation, and a generalized asymptotic expansion, similar to that adopted for the harmonic wave (§4), has to be employed in order to obtain a uniformly valid solution.

7. Removal of resonance and derivation of uniformly valid solution

The behaviour of the regular perturbation series suggests the existence of a phase shift of the wavenumber of the resonant waves, which depends on ϵ. Hence, as in §4 we shall assume that ϕ has a uniformly valid generalized expansion

$$\phi(x,y,t;\,\epsilon) = \epsilon\phi_{1g}(x,y,t;\,\epsilon)+\epsilon^2\phi_{2g}(x,y,t;\,\epsilon)+\dots, \tag{71}$$

with
$$\phi_{jg}(x,y,t;\,\epsilon) = \sum_{m=-\infty}^{m=\infty} \varphi_{j,mg}(x,y;\,\epsilon)\,e^{im\omega t}, \tag{72}$$

replacing (9) and (18) respectively. We limit our study here to deriving in detail the term $\varphi_{1,1g}$. Along the lines of §4 we assume that $\tilde{\varphi}_{1,1g}(k,0)$ can be written as

$$\tilde{\varphi}_{1,1g}(k,0) = \frac{\tilde{p}_{1,1}(k)}{A_{1,1g}(k,\omega;\,\epsilon)} \tag{73}$$

instead of (54), so that
$$\lim_{\epsilon\to 0} A_{1,1g}(k,\omega;\,\epsilon)= A_{1,1}(k,\omega). \tag{74}$$

On the basis of (74) and §4 it is assumed that $A_{1,1g}$ has the structure

$$A_{1,1g}(k,\omega;\,\epsilon)= A_{1,1}(k,\omega)+d_1(\epsilon,\omega);\ \lim_{\epsilon\to 0} d_1(\epsilon,\omega) = 0. \tag{75}$$

The generalized dispersion relation

$$A_{1,1}(k,\omega)+d_1 = 0 \tag{76}$$

has in the neighbourhood of $k = k_c$ the two roots

$$k_{1g}^{(1)} = k_c+\delta\omega+(\delta\omega+d_1)^{\frac{1}{2}},\quad k_{1g}^{(2)} = k_c+\delta\omega-(\delta\omega+d_1)^{\frac{1}{2}}, \tag{77}$$

so that
$$\frac{1}{A_{1,1g}(k,\omega;\,\epsilon)} = \frac{1}{2(\delta\omega+d_1)^{\frac{1}{2}}}\left(\frac{1}{k-k_{1g}^{(2)}}-\frac{1}{k-k_{1g}^{(1)}}\right)\quad (k>0). \tag{78}$$

To determine d_1 we have to carry out the same calculations as in §5, with $A_{1,1}$ replaced by $A_{1,1g}$. Furthermore, we retain now the most-singular third-order term of (29) and the first-order one in the same equation, since the expansion is not uniform near $\omega = \omega_c$. Thus the starting point is now

$$L_1(\varphi_{1,1g})+L_3(\varphi_{1,-1g},\varphi_{1,1g},\varphi_{1,1g})+L_3(\varphi_{1,1g},\varphi_{1,-1g},\varphi_{1,1g})$$
$$+L_3(\varphi_{1,1g},\varphi_{1,1g},\varphi_{1,-1g}) = p_{1,1}(x), \tag{79}$$

replacing both (21) and (29). The Fourier transform of (79) yields, similarly to (34) and (63),

$$A_{1,1}(k,\omega)\,\tilde{\varphi}_{1,1g}(k,0)-\epsilon^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[a_{3,1}^*(k,\lambda,\nu)\,\tilde{\varphi}_{1,1g}(\nu,0)\,\tilde{\varphi}_{1,1g}(\lambda-\nu,0)$$
$$\times\,\tilde{\varphi}_{1,1g}(k-\lambda,0)]\,d\lambda\,d\nu = \tilde{p}_{1,1}(k). \tag{80}$$

This nonlinear integral equation replaces (34) for $j = 1$, $m = 1$, which could be recovered by a regular ϵ-power series expansion of (80). Equation (80) is generally intractable, but it can be solved easily at leading order in $\delta\omega^{-1}$. Indeed, by substituting $\tilde{\varphi}_{1,1g}$ (73) in (80) we obtain

$$\frac{A_{1,1}(k,\omega)\,\tilde{p}_{1,1}(k)}{A_{1,1g}(k,\omega;\epsilon)}-\epsilon^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{b_{3,1}^*(k-\lambda,\lambda-\nu,\nu)}{A_{1,1g}(\nu,\omega;\epsilon)\,A_{1,1g}(\lambda-\nu,\omega;\epsilon)\,A_{1,1g}(k-\lambda,\omega;\epsilon)}\,d\lambda\,d\nu$$
$$\equiv \tilde{p}_{1,1}(k), \tag{81}$$

where $b^*_{3,1}$ is given by (66). Carrying out the integration in (81) exactly as in §6, and retaining the most-singular term of the type (67) yields

$$\frac{A_{1,1}(k,\omega)\,\tilde{p}_{1,1}(k)}{A_{1,1g}(k,\omega;\,\epsilon)}-\frac{\epsilon^2\pi^2 b^*_{3,1}(-k_c,k_c,k_c)}{(\delta\omega+d_1)\,A_{1,1g}(k,\omega;\,\epsilon)}\equiv \tilde{p}_{1,1}(k). \tag{82}$$

It is seen that the identity (82) is satisfied at leading-order in $\delta\omega^{-1}$ and in a neighbourhood $k-k_c = O(\delta\omega^{\frac{1}{2}})$ if

$$A_{1,1}(k,\omega)-\frac{\epsilon^2\pi^2 b^*_{3,1}(-k_c,k_c,k_c)}{(\delta\omega+d_1)\,\tilde{p}_{1,1}(k_c)}\equiv A_{1,1g}(k,\omega;\,\epsilon). \tag{83}$$

By using (75) the unknown d_1 is determined uniquely in the neighbourhood of $\omega=\omega_c$ from the quadratic equation ensuing from (83):

$$d_1 = -\frac{\epsilon^2\pi^2 b^*_{3,1}(-k_c,k_c,k_c)}{(\delta\omega+d_1)\,\tilde{p}_{1,1}(k_c)}. \tag{84}$$

Equation (84) yields

$$d_1 = -\tfrac{1}{2}\delta\omega+\frac{1}{2}\left[\delta\omega^2-\frac{4\epsilon^2\pi^2 b^*_{3,1}(-k_c,k_c,k_c)}{\tilde{p}_{1,1}(k_c)}\right]^{\frac{1}{2}}. \tag{85}$$

The sign in (85) has been selected so that for $\epsilon=0$ and $\delta\omega>0$, $A_{1,1g}\to A_{1,1}$. The quantity $b^*_{3,1}/\tilde{p}_{1,1}$ appearing in (85) is easily obtained by substituting

$$a^*_{3,1}(-k_c,k_c,k_c) = -2k_c^4/\pi$$

from (64) in (66) and (61), yielding

$$\frac{b^*_{3,1}(-k_c,k_c,k_c)}{\tilde{p}_{1,1}(k_c)} = -\frac{1}{\pi 2^7}\tilde{p}_{1,1}(k_c)\,\tilde{p}_{1,1}(-k_c). \tag{86}$$

Since $\tilde{p}_{1,1}(k_c)=\bar{\tilde{p}}_{1,1}(-k_c)$ we find that the right-hand side of (86) is an essentially negative real quantity.

Substituting (86) into (78) yields

$$\begin{aligned}\frac{1}{A_{1,1g}(k,\omega,\epsilon)} &= -\frac{1}{k^{(1)}_{1,1g}-k^{(2)}_{1,1g}}\left(\frac{1}{k-k^{(1)}_{1,1g}}-\frac{1}{k-k^{(2)}_{1,1g}}\right)\\ &= \frac{1}{[2\delta\omega+2(\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{\frac{1}{2}}}\left(\frac{1}{k-k^{(2)}_{1,1g}}-\frac{1}{k-k^{(1)}_{1,1g}}\right),\end{aligned} \tag{87}$$

where
$$\sigma^2 = \frac{\epsilon^2\pi\tilde{p}_{1,1}(k_c)\,\tilde{p}_{1,1}(-k_c)}{2^5},\quad d_1 = -\tfrac{1}{2}\delta\omega+\tfrac{1}{2}(\delta\omega^2+\sigma^{\frac{1}{2}})^{\frac{1}{2}}. \tag{88}$$

Equations (73) and (87) give $\tilde{\varphi}_{1,1g}(k,0)$ as a uniformly valid first-order solution for any $\omega\leqslant\omega_c$ and $\epsilon=o(1)$. Inversion of $\tilde{\varphi}_{1,1g}$ (73) therefore gives, for fixed x, y and for the resonant terms that stem from the semi-residues at $k^{(1)}_{1g}$ and $k^{(2)}_{1g}$ near $k=k_c$,

$$\varphi^*_{1,1g}(x,y) = -\frac{2^{\frac{3}{2}}\pi i\,\tilde{p}_{1,1}(k_c)}{[\delta\omega+(\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{\frac{1}{2}}}\,e^{-ik_cx}\,e^{k_cy}, \tag{89}$$

while the remaining terms, i.e. the semi-residues at $k=k^{(3)}_1$, $k^{(4)}_1$ and the principal-value integral of (57) can be taken precisely as in (57), as they are uniform near $\omega=\omega_c$. It is seen that for $\omega=\omega_c$, i.e. $\delta\omega=0$, we have

$$\epsilon\varphi^*_{1,1g}(x,y) = -\frac{\epsilon^{\frac{1}{2}}i\pi^{\frac{3}{4}}\,2^{\frac{11}{4}}\,\tilde{p}_{1,1}(k_c)\,e^{\frac{1}{4}(y-ix)}}{[\tilde{p}_{1,1}(k_c)\,\tilde{p}_{1,1}(-k_c)]^{\frac{1}{4}}}. \tag{90}$$

Hence $\varphi^*_{1,1g}$ and $\varphi^*_{1,-1g} = \bar{\varphi}^*_{1,1g}$ are finite at $\omega = \omega_c$, and the dominant term of the first-order solution

$$\epsilon\phi^*_{1g} = \epsilon\varphi^*_{1,1g}e^{i\omega t} + \epsilon\varphi^*_{1,-1g}e^{-i\omega t} \tag{91}$$

is no longer resonant there. The ordering of the perturbation series is changed, however. Whereas in the region of uniformity of the regular perturbation series, as well as for the remaining terms of $\epsilon\phi_{1,g}$, the order is $O(\epsilon)$, in the region of non-uniformity it becomes $O(\epsilon^{\frac{1}{2}})$. It should be noted that the generalized solution, although derived for k in the neighbourhood of k_c, is valid for any k, and carries out a continuous transition between the two zones, recalling a composite matched asymptotic solution.

By the same token, the order of the resonant term of $\varphi_{2,2g}$ at ω_c becomes $O(\epsilon^{\frac{3}{2}})$ rather than the $O(\epsilon^2/\delta\omega^{\frac{1}{2}})$ of $\varphi_{2,2}$. Finally, the third-order terms not accounted for in (81), i.e. $\varphi_{3,3}$ and part of $\varphi_{3,1}$, behave under the generalized expansion like $O(\epsilon^{\frac{3}{2}})$ rather than $O(\epsilon^3/\delta\omega^{\frac{3}{2}})$ near $\omega = \omega_c$. Hence the generalized perturbation series has been rendered uniform by the phase shift of the resonant waves, and $\epsilon\phi_{1,g}$ is asymptotic to ϕ for any ω and $\epsilon = o(1)$.

8. Illustration of results: drag and lift of an oscillating doublet

The results obtained in §7 are illustrated now by computing the drag and lift forces experienced by an x-oriented oscillatory point doublet of total output ϵ lying at a dimensionless depth h below the undisturbed free-surface.

The function $\tilde{p}_{1,1}(k)$ is given in this special case by

$$\tilde{p}_{1,1}(k) = -\widetilde{\frac{\partial\psi_{1,1}(x,y)}{\partial y}}\bigg|_{y=0} = \frac{ik}{(2\pi)^{\frac{1}{2}}}e^{-|k|h}, \tag{92}$$

while by (88) and by substituting $k = k_c = 0{\cdot}25$,

$$\sigma^2 = \epsilon^2\pi\, 2^{-5}\,\tilde{p}_{1,1}(k_c)\,\tilde{p}_{1,1}(-k_c) = \epsilon^2\, 2^{-10}\, e^{-\frac{1}{2}h}. \tag{93}$$

Thus the potential $\varphi_{1,1g}$ is given with the aid of (87) and (89) by the following expression, valid for $\delta\omega = \omega_c - \omega > 0$ and for fixed x, y:

$$\varphi_{1,1g} = \frac{1}{2^{\frac{3}{2}}[\delta\omega + (\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{\frac{1}{2}}}[k^{(1)}_{1g}\exp\{-ik^{(1)}_{1g}x + |k^{(1)}_{1g}|(y-h)\} + k^{(2)}_{1g}\exp\{-ik^{(2)}_{1g}x + |k^{(2)}_{1g}|(y-h)\}] + \frac{i}{2\pi}\mathop{{\int\!\!\!\!\!\!-}}\nolimits^{\infty}_{-\infty}{}^{\!\!2}_{\!\!1}\frac{k\exp\{-ikx+|k|(y-h)\}}{A_{1,1}(k,\omega)}dk. \tag{94}$$

The last term of (94), standing for a principal value at the poles $k^{(1)}_{1g}$, $k^{(2)}_{1g}$ (77), is regular for $\delta\omega \to 0$. The singular first term of (94) has to be deleted for $\delta\omega < 0$.

The first-order generalized potential is therefore given by

$$\phi_{1g}(x,y,t,\epsilon) = [2\delta\omega + 2(\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{-\frac{1}{2}}\{k^{(1)}_{1g}\exp\{|k^{(1)}_{1g}|(y-h)\}\cos(k^{(1)}_{1g}x - \omega t) + k^{(2)}_{1g}\exp\{|k^{(2)}_{1g}|(y-h)\}\cos(k^{(2)}_{1g}x-\omega t)\} + \frac{i}{2\pi}\mathop{{\int\!\!\!\!\!\!-}}\nolimits^{\infty}_{-\infty}{}^{\!\!2}_{\!\!1}\left[\frac{\exp\{-i(kx-\omega t)\}}{A_{1,1}(k,\omega)} + \frac{\exp\{i(kx-\omega t)\}}{\bar{A}_{1,1}(k,\omega)}\right]k\exp\{|k|(y-h)\}\,dk. \tag{95}$$

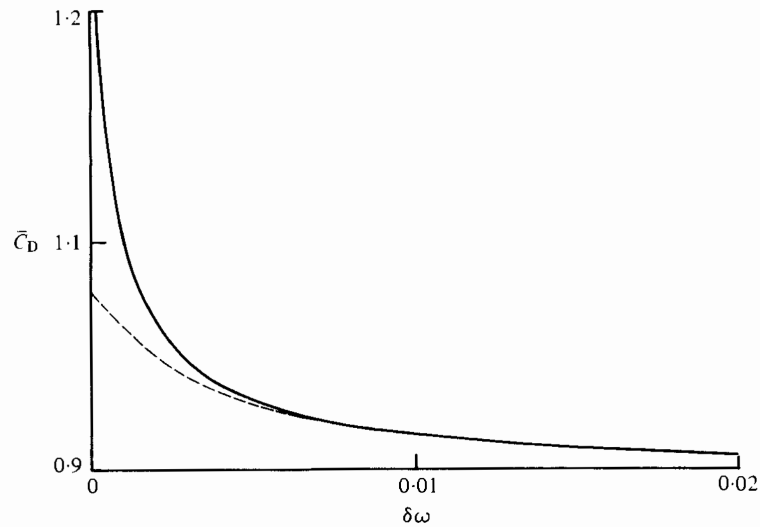

FIGURE 3. The dependence of the DC component of the drag coefficient upon $\delta\omega = \omega_c - \omega$ for $\epsilon = 0{\cdot}1$ and $h = 0{\cdot}1$: ——, the linearized solution given by (98) with $\sigma = 0$; – – – –, the generalized solution given by (98) and (93).

The coefficients of drag and lift experienced by the doublet are readily obtained by employing the Lagally theorem:

$$C_D = \frac{2gD'}{\rho U'^4} = -\pi\epsilon^2 \cos\omega t \left.\frac{\partial^2 \phi_{1g}}{\partial x^2}\right|_{x=0,\, y=-h}, \tag{96}$$

$$C_L = \frac{2gL'}{\rho U'^4} = -\pi\epsilon^2 \cos\omega t \left.\frac{\partial^2 \phi_{1g}}{\partial x\, \partial y}\right|_{x=0,\, y=-h}. \tag{97}$$

Substitution of (95), after extracting the contributions of the semi-residues at $k_1^{(3)}$, $k_1^{(4)}$ (40), yields for $\delta\omega > 0$

$$\begin{aligned} C_D/\pi\epsilon^2 = {} & [2\delta\omega + 2(\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{-\frac{1}{2}} \{k_{1g}^{(1)^3} \exp\{-2|k_{1g}^{(1)}|h\} + k_{1g}^{(2)^3} \exp\{-2|k_{1g}^{(2)}|h\} \cos^2\omega t \\ & + (k_1^{(4)} - k_1^{(3)})^{-1} [k_1^{(4)^3} \exp\{-2|k_1^{(4)}|h\} - k_1^{(3)^3} \exp\{-2|k_1^{(3)}|h\}] \cos^2\omega t \\ & + (4\pi)^{-1} \sin 2\omega t \, \fint_{-\infty}^{\infty} k^3 \left[\frac{1}{A_{1,1}(k,\omega)} - \frac{1}{A_{1,1}(-k,\omega)}\right] \exp\{-2|k|h\}\, dk, \end{aligned} \tag{98}$$

$$\begin{aligned} C_L/\pi\epsilon^2 = {} & \tfrac{1}{2}[2\delta\omega + 2(\delta\omega^2+\sigma^2)^{\frac{1}{2}}]^{-\frac{1}{2}} \{|k_{1g}^{(1)}|^3 \exp\{-2|k_{1g}^{(1)}|h\} + |k_{1g}^{(2)}|^3 \exp\{-2|k_{1g}^{(2)}|h\} \sin 2\omega t \\ & - \tfrac{1}{2}(k_1^{(4)} - k_1^{(3)})^{-1} [|k_1^{(4)}|^3 \exp\{-2|k_1^{(4)}|h\} - |k_1^{(3)}|^3 \exp\{-2|k_1^{(3)}|h\}] \sin 2\omega t \\ & + (2\pi)^{-1} \cos^2\omega t \, \fint_{-\infty}^{\infty} |k|^3 \left[\frac{1}{A_{1,1}(k,\omega)} - \frac{1}{A_{1,1}(-k,\omega)}\right] \exp\{-2|k|h\}\, dk. \end{aligned} \tag{99}$$

The first terms of (98) and (99) have to be deleted for $\delta\omega < 0$. The expressions for σ^2, $k_{1g}^{(1)}$, $k_{1g}^{(2)}$, $k_1^{(3)}$ and $k_1^{(4)}$ are given by (93), (77) and (40) respectively, in terms of ω, h and ϵ. If one substitutes $\sigma = 0$ in (98) and (99) the usual expressions obtained from the linearized solution ϕ_1 are recovered.

It is important to note that the linearized classical solution predicts a DC component of the drag and an AC component of the lift singular like $O(\epsilon^2/\delta\omega^{\frac{1}{2}})$ as $\delta\omega \to 0$.

On the other hand the generalized uniformly valid solution, as given by (98) and (99), implies that the drag and the lift are not singular at the resonant frequency, and in fact are $O(\epsilon^{\frac{3}{2}})$ as $\delta\omega \to 0$. Their maximum values occur at $\delta\omega = 0$, and are given by

$$\begin{Bmatrix} C_{\mathrm{Dmax}} \\ C_{\mathrm{Lmax}} \end{Bmatrix} = \pi 2^{-3} \epsilon^{\frac{3}{2}} e^{-\frac{3}{8}h} \begin{Bmatrix} \sin^2 \omega t \\ \sin \omega t \cos \omega t \end{Bmatrix} + O(\epsilon^{\frac{5}{2}}). \tag{100}$$

The dependence of the DC component of the drag coefficient (98) upon ω for $\epsilon = 0{\cdot}1$ and $h = 0{\cdot}1$ has been represented for the purpose of illustration in figure 3.

9. Extension of results to three-dimensional flows

The analysis of three-dimensional flows, namely the case of moving and oscillating distributions, can be carried out in a similar manner, and only a few results showing the principles will be given here. The investigation of the double Fourier transform of the linearized solution reveals that among the free waves that travel in various directions in the horizontal plane, the amplitude of the transverse waves is the one that becomes unbounded for ω tending to ω_c from below. Furthermore, the first-order potential $\epsilon\phi_1(x, y, 0, t)$ is singular like $\epsilon \ln(\delta\omega)$ as $\delta\omega \to 0$ (Dagan & Miloh 1981). Again, similarly to the two-dimensional case, the third-order terms are the most resonant, and they make the perturbation power-series expansion non-uniform for $\delta\omega \to 0$. The solution can be rendered uniform by the same procedure as in §7, namely by deriving a generalized first-order solution ϕ_{1g} comprising a phase-shift of the far free waves. The final result is that $\epsilon\phi_{1g}$ remains finite for $\delta\omega \to 0$ and behaves like $\epsilon \ln \epsilon$. By the same token the lift and drag acting on an oscillating doublet are shown to be of order $\epsilon^2 \ln \epsilon$ for $\delta\omega \to 0$. The application of these results to moving pressure distribution or ship-like bodies will be discussed elsewhere.

10. Summary and conclusions

The present study has shown that the resonant behaviour of the free waves generated by oscillating singularities moving near a free surface can be removed by maintaining in the same equation the first-order as well as the most-singular contribution of the third-order term. The generalized solution thus obtained is similar to the usual one, except that the two resonant poles, roots of the modified dispersion equation, are displaced. The physical interpretation of the removal of the resonance is that the third-order nonlinear interaction of the free-surface waves causes a shift in the wave-number and the group velocity. The group velocity of the resonant free waves tends for $\delta\omega \to 0$ to the translatory velocity of the singularity in the regular first-order solution. It is shifted by quantities of order ϵ in the two-dimensional case and similarly for the transverse waves in three-dimensional flows. Thus the energy imparted by the oscillating body to the fluid can move away from it, and the amplitude of the free waves as well as the forces acting on the body remain finite. These results are illustrated by computing the lift and drag forces experienced by a horizontal doublet moving parallel and below a free surface.

REFERENCES

BECKER, T. 1958 Das Wellenbild einer unter der oberflache eines stromes schwerer flussigkict pulsierender quelle. *Z. angew. Math. Mech.* **38**, 391–399.

DAGAN, G. & MILOH, T. 1981 Flow past oscillating bodies at resonant frequency. In *Proc. 13th Symp. on Naval Hydrodynamics, Tokyo*, pp. 355–369.

DEBNATH, L. & ROSENBLAT, S. 1969 The ultimate approach to the steady state in the generation of waves on a running stream. *Quart. J. Mech. Appl. Math.* **22**, 221–233.

DOCTORS, L. J. 1978 Hydrodynamic power radiated by a heaving and pitching air-cushion vehicle. *J. Ship Res.* **22**, 67–79.

EGGERS, K. 1957 Uber das Wellenbild einer pulsierenden Storung in Translation. *Schiff und Hafen* **11**, 874–878.

EUVRAND, D., JAMI, A., MORICE, C. & OUSSET, Y. 1977 Calcul numerique des oscillations d'un navire engendrees par la houle. *J. Méc.* **16**, 281–326.

HANAOKA, T. 1957 Theoretical investigation concerning ship motion in regular waves. In *Proc. Symp. on the Behaviour of Ships in a Seaway, Netherland Ship Model Basin.*

HASKIND, M. D. 1954 On the motion with waves of heavy fluid (in Russian). *Prikl. Mat. Mekh.* **18**, 15–26.

LIGHTHILL, M. J. 1960 Studies on magneto-hydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc. Lond.* A **252**, 397–430.

NEWMAN, J. N. 1959 The damping and wave resistance of a pitching and heaving ship. *J. Ship Res.* **3**, 1–19.

NEWMAN, J. N. 1971 Third-order interaction in Kelvin ship–waves systems. *J. Ship Res.* **15**, pp. 1–10.

PHILLIPS, O. M. 1969 *The Dynamics of the Upper Ocean*, 2nd edn. Cambridge University Press.

TAYLER, A. B. & VAN DEN DRIESSCHE, P. 1974 Small amplitude surface waves due to a moving source. *Quart. J. Mech. Appl. Math.* **27**, 317–345.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. In *Handbuch der Physik* (ed. S. Flugge), vol. 9, pp. 446–778. Springer.

WU, T. Y. 1957 Water waves generated by the translatory and oscillatory surface disturbance. *Calif. Inst. of Tech. Engng Div. Rep.* no. 85–3.